# Math 31 – Homework 5

Due Friday, August 2

**Note:** Any problem labeled as "show" or "prove" should be written up as a formal proof, using complete sentences to convey your ideas.

## Easier

**1.** Determine if each mapping is a homomorphism. State why or why not. If it is a homomorphism, find its kernel, and determine whether it is one-to-one and onto.

(a) Define $\varphi : \mathbb{Z} \to \mathbb{R}$ by $\varphi(n) = n$. (Both are groups under addition here.)

(b) Let $G$ be a group, and define $\varphi : G \to G$ by $\varphi(a) = a^{-1}$ for all $a \in G$.

(c) Let $G$ be an *abelian* group, and define $\varphi : G \to G$ by $\varphi(a) = a^{-1}$ for all $a \in G$.

(d) Let $G$ be a group, and define $\varphi : G \to G$ by $\varphi(a) = a^2$ for all $a \in G$.

**2.** Consider the subgroup $H = \{i, m_1\}$ of the dihedral group $D_3$. Find all the left cosets of $H$, and then find all of the right cosets of $H$. Observe that the left and right cosets do not coincide.

**3.** Find the cycle decomposition and order of each of the following permutations.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 7 & 6 & 9 & 8 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 3 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 6 & 7 & 4 \end{pmatrix}$

**4.** Determine whether each permutation is even or odd.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 5 & 1 & 3 & 7 & 8 & 9 & 6 \end{pmatrix}$

(b) $(1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9)$

(c) $(1\ 2\ 3\ 4\ 5\ 6)(1\ 2\ 3\ 4\ 5\ 7)$

(d) $(1\ 2)(1\ 2\ 3)(4\ 5)(5\ 6\ 8)(1\ 7\ 9)$

**5.** Let $G$ and $G'$ be groups, and suppose that $|G| = p$ for some prime number $p$. Show that any group homomorphism $\varphi : G \to G'$ must either be the trivial homomorphism or a one-to-one homomorphism.

## Medium

**6.** [Saracino, #12.13 modified] Let $\varphi : G \to G'$ be a group homomorphism. If $G$ is abelian and $\varphi$ is onto, prove that $G'$ is abelian.

**7.** [Saracino, #12.3 and 12.20 modified] Let $G$ be an abelian group, $n$ a positive integer, and define $\varphi : G \to G$ by $\varphi(x) = x^n$.

(a) Show that $\varphi$ is a homomorphism.

(b) Suppose that $G$ is a finite group and that $n$ is relatively prime to $|G|$. Show that $\varphi$ is an automorphism of $G$.

**8.** [Saracino, #12.33 modified] Let $V_4 = \{e, a, b, c\}$ denote the Klein 4-group. Since $|V_4| = 4$, Cayley's theorem tells us that $V_4$ is isomorphic to a subgroup of $S_4$. In this problem you will apply techniques from the proof of the theorem to this specific example in order to determine which subgroup of $S_4$ is matched up to $V_4$.

Suppose we label the elements of the Klein 4-group using the numbers 1 through 4, in the following manner:

$$\begin{matrix} e & a & b & c \\ 1 & 2 & 3 & 4 \end{matrix}$$

Now multiply every element by $a$ in order, i.e.,

$$\begin{matrix} e & a & b & c \\ 1 & 2 & 3 & 4 \end{matrix} \quad \longrightarrow \quad \begin{matrix} a & e & c & b \\ 2 & 1 & 4 & 3 \end{matrix}$$

Then multiplication by $a$ determines a permutation of $V_4$ (by the proof of Cayley's theorem). This corresponds to an element of $S_4$ via the labels that we have given the elements of $V_4$. Do this for every element $x$ of $V_4$. That is, write down the permutation in $S_4$ (in cycle notation) that is obtained by multiplying every element of $V_4$ by $x$.

## Hard

**9.** [Saracino, #10.32 modified] Let $G$ be a group with identity element $e$, and let $X$ be a set. A **(left) action of $G$ on $X$** is a function $G \times X \to X$, usually denoted by

$$(g, x) \mapsto g \cdot x$$

for $g \in G$ and $x \in X$, satisfying:

1. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in G$ and all $x \in X$.
2. $e \cdot x = x$ for all $x \in X$.

Intuitively, a group action assigns a permutation of $X$ to each group element. (You will explore this idea in part (d) below.)

Finally, there are two important objects that are affiliated to any group action. For any $x \in X$, the **orbit of $x$ under $G$** is the subset

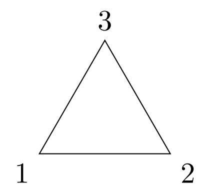
$$\operatorname{orb}(x) = \{g \cdot x : g \in G\}$$

of $X$, and the **stabilizer of $x$** is the subset

$$G_x = \{g \in G : g \cdot x = x\}$$

of $G$.

(a) (Warm up.) We have already seen that it is possible to view the elements of the dihedral group $D_3$ as permutations of the vertices of a triangle, labeled as below:




Thus $D_3$ acts on the set $X = \{1, 2, 3\}$ of vertices by permuting them. Determine the orbit and stabilizer of each vertex under this action.

(b) (Another example.) Let $G$ be a group, let $X = G$, and define a map $G \times X \to G$ by

$$(g, x) \mapsto g \cdot x = gx$$

for all $g \in G$ and $x \in X$, i.e., the product of $g$ and $x$ as elements of $G$. Verify that this defines a group action of $G$ on itself. (This action is called **left translation**.) Given $x \in X = G$, what are $\operatorname{orb}(x)$ and $G_x$?

(c) Prove that for every $x \in X$, the stabilizer $G_x$ is a subgroup of $G$.

(d) Given a fixed $g \in G$, define a function $\sigma_g : X \to X$ by

$$\sigma_g(x) = g \cdot x.$$

Show that $\sigma_g$ is bijective, so $\sigma_g$ defines a permutation of $X$. [Compare this to the proof of Cayley's theorem.]

(e) Recall that $S_X$ denotes the group of permutations of $X$ under composition. Define a function $\varphi : G \to S_X$ by

$$\varphi(g) = \sigma_g$$

for all $g \in G$. Prove that $\varphi$ is a homomorphism. [Note: The proof of Cayley's theorem is a special case of this phenomenon, with $G$ acting on itself by left translation.]

Parts (d) and (e) above show that a group action gives an alternative way of viewing a group as a collection of symmetries (or permutations) of some object. Cayley's theorem provides a specific example, where a group is viewed as a collection of permutations of itself. Group actions provide one of the most interesting ways in which groups are used in practice.